6 Continuous Random Variables

Recall that discrete random variables were used to assign numbers (usually counts of events) to the outcomes in a finite or countably infinite sample space.

If an experiment consists of measuring some quantity that can have any real value, e.g. mass, time, length, ..., then that quantity is a value of a continuous random variable X.

6.1 Probability Density Functions

For a continuous random variable, X, we are interested in the probability that X will take values on an *interval* rather than one specific value.

> **Definition:** For a continuous random variable, X, the probability distribution is described by a probability density function (p.d.f.), $f(x)$, satisfying:
>
> 1. $f(x) \geq 0$ for all x in $\mathbb{R}$
> 2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
> 3. $P(a \leq X \leq b) = \int_a^b f(x)\, dx$

We note that

$$P(x) = 0$$

for any constant x, and as a result,

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = \int_a^b f(x)\, dx.$$

Example 6.1. The p.d.f. for a continuous random variable, X, is

$$f(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 < x \leq 2 \\ \frac{1}{4} & \text{if } 2 < x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Find:

(a) $P(X = 2.2)$

(b) $P(1 \leq X \leq 3)$

(c) $P(1 < X < 3)$

(d) $P(X > 1.2)$

(e) $P(X < 0.6)$

Example 6.2. Find the value of k that makes $f(x)$ a valid p.d.f.:

$$f(x) = \begin{cases} kx^7 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 6.3. [2, p. 153] The mileage (in thousands of miles) that car owners get with a certain kind of tire is a continuous random variable having the p.d.f.

$$f(x) = \begin{cases} \frac{1}{20}e^{-x/20} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the probabilities that one of these tires will last

(a) Exactly 10,000 miles.

(b) At most 10,000 miles.

(c) Anywhere from 16,000 to 24,000 miles.

(d) At least 30,000 miles.

Example 6.4. [2, p. 153] In a certain city, the daily consumption of electric power (in millions of kilowatt hours) is a continuous random variable with the p.d.f.

$$f(x) = \begin{cases} \frac{1}{9}xe^{-x/3} & x > 0 \\ 0 & x \le 0 \end{cases}$$

If the city's power plant has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?

6.2 Mean and Standard Deviation

For discrete random variables, we used sums to calculate the mean and SD. Summing over a continuous domain requires integration.

Definition: For a continuous random variable, X, with p.d.f. $f(x)$,

- the mean or expected value of X is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

- the variance of X is

$$\sigma^2 = E(X^2) - \mu^2, \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

- and the standard deviation (SD) of X is

$$\sigma = \sqrt{\sigma^2}$$

Example 6.5. Find the expected value (aka the average) and SD of the daily consumption of electric power in Example 6.4.

Example 6.6. [2, p. 172] In a certain municipality, the proportion of highway sections requiring repairs in any given year is a continuous random variable with p.d.f.

$$f(x) = \begin{cases} 12x^2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Graph $f(x)$.

(b) Find the expected proportion of highway sections requiring repairs in any given year.

(c) Find the probability that at most half of the highway sections will require repairs in any given year.

6.3 Two Special Distributions

In this section we will define two special forms of p.d.f.s. In the next section we will study the *Normal Distribution*, which has the most commonly used p.d.f.

Definition: A continuous random variable is a uniform random variable if its p.d.f. has the form

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Example 6.7. [2, p. 165] Suppose that the wheel of a train has radius r and that x is the location of a point on its outer edge measured (in cm) along the circumference from some reference point 0. When the brakes are applied, some point on the wheel will make sliding contact with the rail, and heavy wear will occur at that point. For repeated applications of the brakes, it would seem reasonable to assume that x is a value of a uniform random variable with p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{2\pi r} & 0 < x < 2\pi r \\ 0 & \text{otherwise} \end{cases}$$

Otherwise the wheel would eventually have flat spots.

(a) Find the mean of this random variable.

(b) Find the variance of this random variable.

Definition: A continuous random variable is a exponential random variable if its p.d.f. has the form

$$f(x) = \begin{cases} ke^{-kx} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for some constant k.

Example 6.8. [2, p. 171] The Poisson distribution has many important applications in queing problems, e.g. The number of aircraft arriving at an airport. If in a Poisson process the average number of arrivals per unit of time is λ, then the waiting time between successive arrivals is an exponential random variable with

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that on average, 3 trucks arrive per hour to be unloaded at a warehouse, and the number of arrivals per hour is described by the Poisson distribution. Therefore the time (in hours) between the arrival of successive trucks to be unloaded at a warehouse is given by the p.d.f.

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that the time between the arrival of successive trucks will be

(a) less than 5 minutes.

(b) at least 45 minutes.

Additional Notes

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