5 Binomial and Poisson Distributions

In this section we will look at the probability distributions of two useful discrete random variables.

5.1 The Binomial Distribution

Many statistical experiments involve repeated trials. Here we are interested in a series of "identical success/failure trials", by which we mean:

1. each trial has two possible outcomes, which are called "success" and "failure" (a success is not necessarily the desirable outcome, it is simply the outcome of interest in the experiment),
2. the probability of success is the same for each trial, and
3. the outcome of a trial is independent of the outcome of any other trial.

A series of trials with these features is often referred to as a *Bernoulli trial*.

Definition: Let $X =$ the number of successes in n identical success/failure trials. Then X has the binomial distribution with

$$P(x) = nCx \cdot p^x \cdot q^{n-x},$$

where p is the probability of success on a single trial, and $q = 1-p$ is the probability of failure on a single trial.

For example, if we consider *all* possible ways to answer the *Surprize Quiz* from Section 4 with a success being a correct answer, then

$$X = \text{ the number of correct answers}$$

has a binomial distribution with $n = 10$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$. So that the probability of getting 4 correct answers is

$$P(4) = 10C4 \cdot \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^6 \approx 0.23.$$

Why is this probability different from the $P(4)$ we calculated for the demonstration?

Example 5.1. [1, p. 53] In genetics, a *dihybrid cross* is a cross between two varieties of the same species that differ in two observed traits. A 1923 genetics article reported a cross between yellow-wrinkled and green-round peas. The variable of interest was

$$X = \text{ the number of YR (yellow and round) peas in a pod.}$$

Mendelian laws of inheritance imply that the probability of an individual pea being YR is $p = \frac{9}{16}$.

(a) Find the probability distribution for pods with 6 peas.

(b) Draw the histogram for the distribution in part (a).

Note: Histograms for binomial distributions are symmetric if and only if $p = \frac{1}{2}$.

Example 5.2. An experiment consists of rolling a standard 6-sided die 13 times. Find the probability of rolling at most two 5s or 6s.

Example 5.3. A solar panel installation company makes the claim that in 90% of their installations, the utility bill is reduced by at least one third. If we interpret this to mean that for any installation, the probability of the utility bill being reduced by at least one third is 0.9, find the probability that the utility bill will be reduced by at least one third in

(a) exactly 9 of the next 10 installations.

(b) at least 8 of the next 10 installations.

A convenient online tool for calculating and graphing binomial distributions is available at `http://homepage.divms.uiowa.edu/~mbognar/applets/bin.html`

5.2 The Poisson Distribution

One of the main features of the binomial distribution is that there are a fixed number of trials. For counts that do not have a natural upper bound given by the number of trials, the Poisson distribution is often used as a model.

> **Definition:** Let X = the number of occurrences of an event in a unit of time or space, with λ = the average number of occurrences of that event in that unit of time or space. Then the Poisson distribution has probabilities given by
>
> $$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

The SHARP EL-531X has a button for the factorial function. To calculate $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, use $\boxed{5} \to \boxed{\text{2nd F}} \to \boxed{4}$ *to select* $n!$ $\to \boxed{=}$.

Example 5.4. Uncle Tom has been crabbing from the Sidney Pier every Sunday for the past 5 years. He has been diligently recording the number of crabs in his trap each week, and found that on average, he catches 4 crabs in an outing. Let X be the number of crabs caught in an outing. We want to find the probability distribution of X.

(a) Why is the binomial distribution not appropriate in this example?

(b) Find the Poisson distribution for $X = 0, 1, 2, \cdots, 10$. Round off each probability to 2 decimal places.

(c) Draw the histogram for the distribution in part (b) .

(d) Find $P(X \leq 10)$. Why is it not 1?

A convenient online tool for calculating and graphing Poisson distributions is available at `http://homepage.divms.uiowa.edu/~mbognar/applets/pois.html`

Example 5.5. It has been found that the average daily traffic volume on a certain quiet street is 7, and can be modelled using the Poisson distribution.

(a) What is the probability that the traffic volume will be at most 5 on a given day?

(b) What is the probability that the traffic volume will at least 5 on a given day?

Example 5.6. Over the past 30 years, there have been an average of 8 days in January with rainfall above 5mm in Victoria. Using a Poisson distribution, find the probability that next January there will be

(a) Exactly 8 days of rainfall above 5mm in Victoria.

(b) Less than 8 days of rainfall above 5mm in Victoria.

(c) More than 8 days of rainfall above 5mm in Victoria.

(d) Repeat (a) to (c) using a binomial distribution and assuming that each January day has a 8/31 chance of having rainfall above 5mm.

Example 5.7. Suppose that the concentration of bacteria in the inner harbour is 3 per 100 mL of water. Use an appropriate distribution to find the probability that there are at most 2 bacteria in a 50 mL sample of water.

Example 5.8. There are an average of 1.8 accidents per week on a certain highway. Use an appropriate distribution to find the probability that there will be at least 4 accidents in the next 2 weeks.

Additional Notes

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